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Bounded interpolations between lattices

Michel Duneau and Christophe Oguey†
Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau Cedex, France

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Abstract. It is shown that, given two arbitrary lattices of equal density in the Euclidean space $\mathbb{R}^n$, a bounded quasi-periodic and piecewise affine vector field $\boldsymbol{v}$ on $\mathbb{R}^n$ (a so-called 'modulation field') can be built so that the second lattice is the image of the first one under the map $x \to x - v(x)$. The proof relies on a factorization lemma for matrices with determinant equal to one. Each factor represents a shear-like transformation of $\mathbb{R}^n$ which, in turn, is closely approximated by a periodic set of 'slips' in the lattice.

1. Introduction and results

In the Euclidean space $E = \mathbb{R}^n$ of dimension n, a lattice L is a discrete subgroup of the translations of E that we shall always assume to have maximal dimension, i.e. L is isomorphic to $\mathbb{Z}^n$. A cell or fundamental domain for L is a closed region of E the L-orbit of which covers E without overlap (the interior of two different cells in the orbit do not intersect). Our main result is the following.

Theorem 1. Given two arbitrary lattices L_a and L_b of equal density in E, there is a map f of E onto itself, hereafter called a modulation, satisfying the following properties:

(i) f is one-to-one on E;

(ii) f is piecewise affine;

(iii) the field $x \to v(x) = x - f(x)$ for x in E is bounded and 'minimally' quasiperiodic, meaning that the dimension of its frequency module is at most $n+1$;

(iv) f maps L_a onto $f(L_a) = L_b$.

We call f the displacement or modulation *map* whereas $v = I - f$ is referred to as the displacement or modulation *field*. After the theorem, both are piecewise affine. The former is close to the identity I, the latter is uniformly bounded. It should be mentioned, however, that such mappings f, satisfying all four conditions of theorem 1, are not unique. Suppose, for instance, that $L_a = L_b$; then besides the trivial solution ($f = I$ and $v = 0$), various equally valid solutions can be constructed by means of periodic permutations of vertices. Nevertheless, for a given upper bound on the displacements, the set of solutions should be finite (condition (iii) is a severe restriction which rules out anarchic solutions).

The proof of the theorem, in section 4, provides an explicit construction of a modulation f and specifies, among other things, the domains where f and v are linear.

† Also at: Université Paris-Sud, F-91405 Orsay, France.

Of course, there always exists a linear map t in GL(E) mapping L_a onto $t(L_a) = L_b$. For instance, such a map is completely determined by specifying two bases, one for L_a and one for L_b. However, there is no uniform bound on the corresponding displacement $x - t(x)$.

We call the *transition matrix* from L_a to L_b the matrix T, with respect to a definite basis of L_a, of any linear map t satisfying $t(L_a) = L_b$. For instance if $L_a = A\mathbb{Z}^n$ and $L_b = B\mathbb{Z}^n$, where A and B are two regular $n \times n$ matrices, the corresponding transition matrix T from L_a to L_b is $T = A^{-1}B$. The ratio of the co-volumes is $|\det T|$. So the densities of the lattices L_a and L_b are equal if, and only if, $\det T = \pm 1$ for any transition matrix T.

The main stage in proving the theorem is a decomposition of a suitably chosen transition matrix into a product of 'shear' matrices. A *shear* matrix $M^{(k)}$, where k is an integer, differs from the identity I only on the off-diagonal elements of the kth row (or $((k-1) \bmod n) + 1$ if k is not in $\{1, \ldots, n\}$):

$$M^{(k)} = \begin{pmatrix} 1 & 0 & \ldots & & \ldots & & 0 \\ * & \ldots & * & 1 & * & \ldots & * \\ 0 & \ldots & & \ldots & & 0 & 1 \end{pmatrix} \leftarrow k\text{th row.}$$

The factorization property is stated in a lemma:

Lemma. If A is a real $n \times n$ matrix with determinant 1, then there are shear matrices $M^{(1)}, \ldots, M^{(n)}, M^{(n+1)}$ (as described above) and a modular matrix $U \in GL(n, \mathbb{Z})$ such that

$$AU = M^{(n+1)}M^{(n)} \ldots M^{(1)}.$$

Generically, i.e for almost all matrix A, one can take $U = I$ in the lemma.

Every shear matrix depends on $n-1$ real parameters. It turns out that these parameters define a vector which is dual to a generator of the frequency module (that is the $\mathbb{Z}$-module which carries the Fourier transform of the displacement field) so that the dimension of the frequency module equals the number of factors in the lemma, namely $n+1$. The number of independent parameters is $(n-1)(n+1)$, which is the 'dimension' n^2-1 of the set of all the lattices with equal, fixed, density.

There are special non-trivial situations where the modulation field $\boldsymbol{v}$ in the theorem is *periodic*. These special cases are characterized by several equivalent conditions:

(i) L_a and L_b admit a common (non-conventional) fundamental domain which actually is a parallelotope;

(ii) there is a transition matrix $T: L_a \to L_b$ such that $T^{-1} = M^{(n)} \ldots M^{(1)}$ (with only n factors) where $M^{(1)}, \ldots, M^{(n)}$ are shear matrices as in the lemma;

(iii) there is a transition matrix T whose inverse has all its first principal minors equal to one.

The present analysis originated in the study of displacive transitions in solid state physics. The most simple instances of displacive transforms occur in modulated crystals; theoretical investigations in this field have been invigorated by the discovery of quasicrystals (Shechtman *et al*) and more 'tricky' types of atomic orderings. Fundamental questions like that of structural stability rely upon characterizing the nature, 'directions' or 'amplitude' of the collective motions atoms can undergo. See Duneau and Oguey (1990) for a recent issue in this area. Similar problems arise in the structural analysis of interfaces in poly-crystalline materials, as well as in the study of martensitic transformations (work in progress).

The problem handled here also concerns the field of space quantization by regular lattices. Essentially, what we are looking for is a multidimensional extension of the 'round' function. Taking the 'round' of every coordinate amounts to quantization of space by squares, but in the simplest instance of quantizing a rotated copy of the reference lattice, several nodes of the copy sometimes fall into the same square; so the one-to-one property does not hold. In the fourth section we explicitly build a bijective and bounded map between the two lattices. When applied to dimension $n = 2$, our construction provides a (reversible) algorithm to turn a bit map image by an arbitrary angle without changing the number of active dots.

Another issue of our result deals with connectivity. Together with the set of nodes one can consider the net of bonds, faces, etc, that is, the whole standard cell complex structure of the lattice. The map f actually maps the cell structure of L_a onto the cell structure of L_b. Locally, this is a direct consequence of a the fact that f coincides with a regular affine map. Moreover it will be clear from the construction of the map that, even if some lattice lines, planes, etc, undergo tearing 'during' the transition, all the torn pieces do match again, though connected in a different order, after the transition.

The article is organized as follows: basic notations and definitions are collected in section 2. The factorization lemma is proved in section 3, which is pure linear algebra. Section 4 provides an explicit construction of a map f and the related field $\boldsymbol{v}$. This is achieved in several stages: applying the factorization lemma to the matrix T of $\mathrm{SL}(n, \mathbb{R})$, which relates the two lattices, we decompose the map into shear matrices. Every shear transform is at a bounded 'distance' from a periodic set of 'slips' (slips preserve the lattice). The difference 'shear–slip' provides a bounded displacement Φ_k, which is the basic step of the construction (section 4.1). The first n stages $k = 1, \ldots, n$ are then composed to form a periodic map Φ sending the initial lattice onto an intermediate one (section 4.2). In many instances this lattice is the final one; in other words, n basic steps are enough and the algorithm stops here. But in generic situations, an additional shear–slip step $k = n + 1$ is required to get the target lattice and the overall mapping f (section 4.3). The quasiperiodicity of $\boldsymbol{v}$ is settled down there. The last section (section 5) is devoted to the (non-generic) cases where n steps suffice: geometrically, the lattices correspond to two different packings of the same polyhedron.

2. Notation and definitions

Throughout the text, E is a n-dimensional Euclidean space endowed with a scalar product $(\cdot, \cdot)$ and an orthonormal basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$; so E is isomorphic to $\mathbb{R}^n$.

The standard scalar product in $\mathbb{R}^n$ is $\langle x, y\rangle = \Sigma_i x_i y_i$ for x, y in $\mathbb{R}^n$, and the canonical basis of $\mathbb{R}^n$ is $(e_1, \ldots, e_n)$: $(e_i)_j = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$.

If $(\boldsymbol{a}) = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ and $(\boldsymbol{b}) = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ are two bases of E, the transformation matrix T: $(\boldsymbol{a}) \to (\boldsymbol{b})$ is defined by the coefficients $\boldsymbol{b}_i = \Sigma_j T_{ji}\boldsymbol{a}_j$ for $i = 1, \ldots, n$. If $\boldsymbol{x} = \Sigma_i x_i\boldsymbol{a}_i = \Sigma_i y_i\boldsymbol{b}_i$ is a point in E with coordinates $x = (x_1, \ldots, x_n)$ in $(\boldsymbol{a})$ and $y = (y_1, \ldots, y_n)$ in $(\boldsymbol{b})$ respectively, the change in coordinates is $x_i = \Sigma_j T_{ij}y_j$ or simply $x = Ty$.

$\mathrm{GL}(n, \mathbb{R})$ is the group of regular $n \times n$ matrices with coefficients in $\mathbb{R}$.

$\mathrm{SL}(n, \mathbb{R})$ is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ of matrices with determinant 1.

$\mathrm{GL}(n, \mathbb{Z})$ is the group of modular matrices. A modular matrix is an integer matrix which is invertible over $\mathbb{Z}$. The determinant of such matrices is necessarily ± 1.

$O(n)$ is the group of orthogonal matrices.

$UT(n, \mathbb{R})$ is the group of upper triangular matrices with all diagonal entries equal to 1.

$LT(n, \mathbb{R})$ is the group of lower triangular matrices with diagonal elements equal to 1.

Both $UT(n, \mathbb{R})$ and $LT(n, \mathbb{R})$ are subgroups of $SL(n, \mathbb{R})$.

For $1 \leqslant i, j \leqslant n$, we let E_{ij} be the matrix with all entries zero but the element on row i and column j which is equal to 1. The set of all the matrices E_{ij} with $1 \leqslant i, j \leqslant n$ is a basis for the vector space of $n \times n$ matrices.

For any k in $\{1, \ldots, n\}$, $\mathscr{H}^{(k)}$ is the linear subspace, generated by $E_{k,1}, \ldots, E_{k,k-1}$, $E_{k,k+1}, \ldots, E_{k,n}$. The elements of $\mathscr{H}^{(k)}$ are called 'row matrices' (of index k), with a hole in diagonal position; they can be written as $H = \Sigma_{j \neq k} c_j E_{k,j}$ with $c_j \in \mathbb{R}$.

$\mathscr{M}^{(k)} = \exp(\mathscr{H}^{(k)})$ is the Abelian subgroup of $SL(n, \mathbb{R})$ of *shear matrices* (of index k); these are matrices of the form $\exp(H)$ for H in $\mathscr{H}^{(k)}$. Since $H^2 = 0$ for any member of $\mathscr{H}^{(k)}$, actually $\exp(\Sigma_{j \neq k} c_j E_{kj}) = I + \Sigma_{j \neq k} c_j E_{kj} = \Pi_{j \neq k} (I + c_j E_{kj})$.

The inverse of a shear matrix is given by $(I + H)^{-1} = I - H$.

3. The factorization lemma

The purpose of this section is to show that there are two bases—one for each lattice—such that the transition matrix can be decomposed into a product of shear matrices.

Once a definite basis has been chosen in each lattice, the problem reduces to pure matrix algebra. We shall write the statements in terms of an arbitrary matrix A of $SL(n, \mathbb{R})$. Changes of bases in L_a (resp L_b) correspond to post- (resp pre-) multiplication by elements of $GL(n, \mathbb{Z})$.

A multi-index in $\{1, \ldots, n\}$ is an ordered subset $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $\{1, \ldots, n\}$. If α and β are two multi-indices of length k in $\{1, \ldots, n\}$, $A[\alpha|\beta]$ denotes the $k \times k$ submatrix of A taken out from A by dropping all rows not in α and all columns not in β and reordering the rows and columns accordingly.

The diagonal subdeterminants (the principal *minors*, for short) of a matrix A are the determinants of the form $\det\{A[\alpha|\alpha]\}$ for any multi-index α in $\{1, \ldots, n\}$.

Lemma 3.1. Let i, j be two integers such that $1 \leqslant i \neq j \leqslant n$ and c be a real number. The principal minors of the matrices A and $A(1 + cE_{ij})$ are equal for all multi-indices α which contain i or do not contain j:

$$i \in \alpha \text{ or } j \notin \alpha \Rightarrow \det\{A(1 + cE_{ij})[\alpha|\alpha]\} = \det\{A[\alpha|\alpha]\}.$$

Proof. Let $B = A(1 + cE_{ij})$ and let $A_{.k}$ and $B_{.k}$ denote the kth columns of A and B respectively. Then $B_{.k} = A_{.k}$ for all $k \neq j$ and $B_{.j} = A_{.j} + cA_{.i}$. Now, if $j \notin \alpha$ the submatrices $A[\alpha|\alpha]$ and $B[\alpha|\alpha]$ are equal and consequently the corresponding minors are also equal; if $j \in \alpha$ and $i \in \alpha$, $\det\{B[\alpha|\alpha]\} = \det\{A[\alpha|\alpha]\}$ since the contribution of $cA_{.i}$ vanishes by skew-symmetry. □

Lemma 3.2. Let A be an $n \times n$ matrix with determinant 1. There exist a modular matrix U and a shear matrix $M \in \mathscr{M}^{(1)}$ such that for all $k = 1, \ldots, n$

$$\det\{MAU[(1, \ldots, k)|(1, \ldots, k)]\} = 1.$$

Proof. The matrix U is a permutation matrix which satisfies the following conditions:

$$\det\{AU[(2,\ldots,k,k+1)|(1,\ldots,k)]\}\neq 0 \qquad \text{for all } k=1,\ldots,n-1.$$

Such a permutation always exits for regular matrices (see Gantmacher 1977, ch II).

Set $A'=AU$ and $B=MA'=(I+\Sigma_{i=2}^n c_iE_{1i})A'$. Notice that the rows of B are related to the rows of A' by $B_{1.}=A'_{1.}+\Sigma_{i=2}^n c_iA'_{i.}$, and $B_{i.}=A'_{i.}$ for $i=2,\ldots,n$.

Let $a'_k=\det\{A'[(1,\ldots,k)|(1,\ldots,k)]\}$ and $b_k=\det\{B[(1,\ldots,k)|(1,\ldots,k)]\}$. Then, by linearity with respect to the first line and skew-symmetry, we have

$$b_n=a'_n$$

$$b_k=a'_k+\sum_{i=2}^{n} c_i\det\{A'[i,2,\ldots,k)|(1,\ldots,k)]\}$$

$$=a'_k+\sum_{i=k+1}^{n} c_i\det\{A'[i,2,\ldots,k)|(1,\ldots,k)]\} \qquad \text{for } k=1,\ldots,n-1.$$

For $k=n$, $b_n=1$ with no restriction on the elements c_k of M.

For $k=n-1$, we have $b_{n-1}=a'_{n-1}+c_n\det\{A'[(n,2,\ldots,n-1)|(1,\ldots,n-1)]\}$. This last determinant does not vanish, by the choice of U. Then the value of c_n can be chosen in order that $b_{n-1}=1$. Now assume that the elements $c_n,\ldots,c_{k+2}$ have been determined such that $b_n,\ldots,b_{k+1}$ are equal to one. In the equation $b_k=a'_k+\Sigma_{i=k+1}^n c_i\det\{A'[(i,2,\ldots,k)|(1,\ldots,k)]\}=1$, the coefficient of c_{k+1} is $\det\{A'[(k+1,2,\ldots,k)|(1,\ldots,k)]\}\neq 0$, leading to a unique solution for c_{k+1}.

Proceeding this way down to c_2 provides the required elements of the matrix M. □

Remark. Generically, i.e. for almost all matrices A, the above property holds with $U=I$. However, in general, different choices of U are possible and, consequently, different global solutions will be obtained. This is the source of the non-uniqueness of solutions.

Lemma 3.3. Let B be a matrix of $\mathrm{SL}(n,\mathbb{R})$ such that $\det\{B[(1,\ldots,k)|(1,\ldots,k)]\}=1$ for all $k=1,\ldots,n$. Then there exist shear matrices $M^{(1)}\in\mathcal{M}^{(1)},\ldots,M^{(n)}\in\mathcal{M}^{(n)}$ such that

$$B=M^{(n)}M^{(n-1)}\ldots M^{(1)}.$$

Proof.

(i) Set $c_j^{(1)}=B_{1j}$ and $M^{(1)}=\exp(\Sigma_{j=2}^n c_j^{(1)}E_{1j})$. By hypothesis, $B_{11}=1$ so that the matrix $B^{(1)}=BM^{(1)-1}=B(1-\Sigma c_j^{(1)}E_{1j})$ is of the form

$$\begin{pmatrix}1 & 0 & \ldots & 0\\ * & \ldots & \ldots & *\\ * & \ldots & \ldots & *\end{pmatrix}.$$

By lemma 3.1, B and $B^{(1)}$ have equal diagonal minors for α of the type $(1,\ldots,k)$. In particular, $B_{22}^{(1)}=1$.

(ii) Set $c_j^{(2)} = B_{2j}^{(1)}$ for $j = 1, 3, \ldots, n$ and $M^{(2)} = \exp(\Sigma_{j\neq 2}\, c_j^{(2)} E_{2j})$. Then $B^{(2)} = B^{(1)} M^{(2)-1}$ is of the form

$$\begin{pmatrix} 1 & 0 & - & - & - & 0 \\ 0 & 1 & 0 & - & - & 0 \\ * & - & - & - & - & * \\ * & - & - & - & - & * \end{pmatrix}.$$

Again, B, $B^{(1)}$ and $B^{(2)}$ have equal principal minors so that $B_{33}^{(2)} = 1$.

(iii) Proceeding this way up to n we get

$$B^{(n)} = B^{(n-1)} M^{(n)-1} = BM^{(1)-1} \ldots M^{(n)-1} = I$$

which proves the lemma. □

Remarks.

(i) Lemma 3.3 essentially is Gauss's diagonalization (triangulation) procedure (Gantmacher 1977).

(ii) Actually, the statement of lemma 3.3 holds with an 'if and only if'. Indeed, it is straightforward to deduce, from lemma 3.1, that a product $B = M^{(n)} \ldots M^{(1)}$ with $M^{(j)} \in \mathcal{M}^{(j)}$ for all $j = 1, \ldots, n$ satisfies $\det\{B[(1, \ldots, k)|(1, \ldots, k)]\} = 1$.

Proposition 3.4 (factorization lemma). Let A be a matrix with determinant equal to 1. There exist a modular matrix U and shear matrices $M^{(1)} \in \mathcal{M}^{(1)}$, $M^{(2)} \in \mathcal{M}^{(2)}, \ldots, M^{(n)} \in \mathcal{M}^{(n)}$ and $M^{(n+1)} \in \mathcal{M}^{(1)}$ such that

$$AU = M^{(n+1)} M^{(n)} \ldots M^{(1)}.$$

Proof. By lemma 3.2, there exist U in $\mathrm{GL}(n, \mathbb{Z})$ and $M^{(n+1)}$ in $\mathcal{M}^{(1)}$ such that $B = (M^{(n+1)})^{-1} AU$ has diagonal minors $\det\{B[(1, \ldots, k)|(1, \ldots, k)]\}$ equal to 1 for all $k = 1, \ldots, n$. Then lemma 3.3 ensures the existence of the matrices $M^{(k)}$ for $k = 1, \ldots, n$ such that

$$(M^{(n+1)})^{-1} AU = M^{(n)} \ldots M^{(1)}.$$ □

Example. Let R be the matrix of a $\pi/4$ rotation in $\mathbb{R}^2$. The decomposition of R reads

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2}-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2}-1 \\ 0 & 1 \end{pmatrix}.$$

4. The piecewise affine modulation

Given two lattices L_a and L_b of equal density, there is, according to proposition 3.4, a basis $(\boldsymbol{a})$ of L_a and a basis $(\boldsymbol{b})$ of L_b such that the inverse S of the transition matrix $T : (\boldsymbol{a}) \to (\boldsymbol{b})$ is a product of shear matrices (S is itself a transition matrix, $S : (\boldsymbol{b}) \to (\boldsymbol{a})$):

$$S = M^{(n+1)} M^{(n)} M^{(n-1)} \ldots M^{(1)}.$$

Let us define a sequence of bases of E by the following chain of transitions:

$$(\boldsymbol{a})=(\boldsymbol{a}^{(0)}) \xrightarrow{M^{(1)^{-1}}} (\boldsymbol{a}^{(1)}) \xrightarrow{M^{(2)^{-1}}} (\boldsymbol{a}^{(2)}) \ldots (\boldsymbol{a}^{(n-1)}) \xrightarrow{M^{(n)^{-1}}} (\boldsymbol{a}^{(n)}) \xrightarrow{M^{(n+1)^{-1}}} (\boldsymbol{a}^{(n+1)})=(\boldsymbol{b}).$$

For any k in $\{1,\ldots,n+1\}$ the matrix $M^{(k)}$ is a shear matrix of the form $I+H^{(k)}$. If $h^{(k)}$ denotes the non-zero row of $H^{(k)}$, the transformation rule may be written as

$$\boldsymbol{a}_i^{(k)}=\sum (M^{(k)^{-1}})_{ji}\boldsymbol{a}_j^{(k-1)}=\boldsymbol{a}_i^{(k-1)}-h_i^{(k)}\boldsymbol{a}_k^{(k-1)} \qquad \text{for } i=1,\ldots,n$$

where $h_k^{(k)}=0$. All the vectors $\boldsymbol{a}_1^{(k-1)},\ldots,\boldsymbol{a}_n^{(k-1)}$ except $\boldsymbol{a}_k^{(k-1)}$ undergo a shift parallel to $\boldsymbol{a}_k^{(k-1)}$.

For any point x of E the column vectors $x^{(0)}, x^{(1)},\ldots,x^{(n)}, x^{(n+1)}$ denote the coordinates of x with respect to the $n+2$ basis $(\boldsymbol{a}^{(0)}), (\boldsymbol{a}^{(1)}),\ldots,(\boldsymbol{a}^{(n+1)})$ respectively. The above changes of basis correspond to the following transformation rules:

$$\begin{aligned} x^{(1)} &= M^{(1)}x^{(0)} \\ &\ldots \\ x^{(n)} &= M^{(n)}x^{(n-1)} \\ x^{(n+1)} &= M^{(n+1)}x^{(n)}. \end{aligned}$$

The overall coordinate transformation from the basis $(\boldsymbol{a}^{(0)})$ to the basis $(\boldsymbol{a}^{(n+1)})$ is thus provided by

$$x^{(n+1)}=M^{(n+1)}M^{(n)}\ldots M^{(1)}x^{(0)}=Sx^{(0)}.$$

For every $k=1,\ldots,n+1$, the transformation rule can be written in terms of the coefficients $h^{(k)}$ as

$$x^{(k)}=x^{(k-1)}+\langle h^{(k)}, x^{(k-1)}\rangle e_k.$$

We will also consider the intermediate lattices $L_a=L^{(0)}, L^{(1)},\ldots,L^{(n)}, L^{(n+1)}=L_b$ generated by the various bases. For any k in $\{0,\ldots,n+1\}$, a point x belongs to $L^{(k)}$ if its coordinates $x^{(k)}=(x_1^{(k)},\ldots,x_n^{(k)})$ with respect to the basis $(\boldsymbol{a}^{(k)})$ are integers.

For generic values of the coefficients $h_i^{(k)}$ two successive lattices $L^{(k-1)}$ and $L^{(k)}$ have an intersection reduced to the one-dimensional lattice $\mathbb{Z}\boldsymbol{a}_k^{(k)}$.

Example. If L_a, L_b are two square lattices, at 45° to each other, the transition matrix is the one mentioned at the end of section 3. The related bases in the plane are depicted in figure 1.

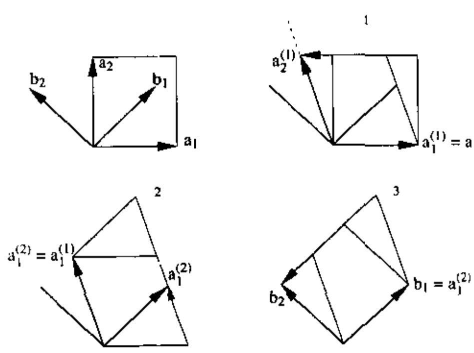

Figure 1. The bases of the four lattices $L^{(0)}$, $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ involved in the algorithm for the example of two square lattices rotated by 45°. This example is generic as far as the number of basic steps is concerned $(n+1=3)$.

We next define a sequence of mappings $\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n+1}$ from E onto E which are close to the identity and which map respectively $L^{(0)}$ onto $L^{(1)}$, $L^{(1)}$ onto $L^{(2)}, \ldots, L^{(n)}$ onto $L^{(n+1)}$.

4.1. *The mappings φ_k (periodic shear-slips)*

First we need a decomposition of the real numbers into integer and fractional parts: $\alpha = \mathrm{Rnd}(\alpha) + r(\alpha)$ where $\mathrm{Rnd}(\alpha) = m$ if $m - \frac{1}{2} \leq \alpha < m + \frac{1}{2}$ with m in $\mathbb{Z}$.

In this section we fix k in $\{1, \ldots, n+1\}$.

The mapping φ_k is defined by $\varphi_k(x) = y$ where the point $x = \Sigma_i x_i^{(k-1)} a_i^{(k-1)}$ and its image $y = \Sigma\, y_i^{(k)} a_i^{(k)}$ are related by the following transformation of the coordinates:

$$y^{(k)} = x^{(k-1)} + \mathrm{Rnd}(\langle h^{(k)}, x^{(k-1)} \rangle) e_k.$$

Using the above expression of $x^{(k)}$ we find that

$$\begin{aligned} y^{(k)} &= x^{(k)} - \langle h^{(k)}, x^{(k-1)} \rangle e_k + \mathrm{Rnd}(\langle h^{(k)}, x^{(k-1)} \rangle) e_k \\ &= x^{(k)} - r(\langle h^{(k)}, x^{(k-1)} \rangle) e_k. \end{aligned}$$

Let us define the vector q_k by $(q_k, x) = \langle h^{(k)}, x^{(k-1)} \rangle$ for all x in E, $x^{(k-1)}$ denoting the coordinates of x in the basis $a^{(k-1)}$. The vector q_k is given by $q_k = \Sigma_j h_j^{(k)} a_j^{(k-1)*}$ in terms of the dual basis $a_j^{(k-1)*}$ of $a_j^{(k-1)}$. Then φ_k has the equivalent definition

$$y = \varphi_k(x) = x - w_k(x)$$

where the displacement field is

$$w_k(x) = r((q_k, x)) a_k^{(k)}.$$

The field w_k is a one-dimensional field directed along $a_k^{(k)}$ (remember that $a_k^{(k-1)} = a_k^{(k)}$) whose amplitude is bounded by $\|a_k^{(k)}\|/2$. Moreover, since $(q_k, a_k^{(k)}) = h_k^{(k)} = 0$, the field w_k does not depend on the component of its argument along $a_k^{(k)}$ (the 'longitudinal' component).

Lemma 4.1. The map φ_k is one-to-one on $\mathbb{R}^n$ and the inverse map is $x \to \varphi_k^{-1}(x) = x + w_k(x)$. Furthermore, φ_k maps the lattice $L^{(k-1)}$ onto $L^{(k)}$: $\varphi_k(L^{(k-1)}) = L^{(k)}$.

Proof. Let $y = \varphi_k(x) = x - r((q_k, x)) a_k^{(k)}$. Then $(q_k, y) = (q_k, x)$ since $(q_k, a_k^{(k)}) = 0$. This implies $x = y + r((q_k, y)) a_k^{(k)}$ which proves that $x \to x + w_k(x)$ is an inverse for φ_k. From the first definition of φ_k, the components $x^{(k-1)}$ of x in $(a^{(k-1)})$ and the components $y^{(k)}$ of $\varphi_k(x)$ in $(a^{(k)})$ differ by integers:

$$y^{(k)} - x^{(k-1)} = \mathrm{Rnd}(\langle h^{(k)}, x^{(k-1)} \rangle) e_k = \mathrm{Rnd}(\langle h^{(k)}, y^{(k)} \rangle) e_k.$$

So the nodes of $L^{(k-1)}$ are mapped into nodes of $L^{(k)}$ and reciprocally. □

The translation symmetries of the displacement field form a larger set than stated so far. Let Q_k be the 'grid', a countable union of hyperplanes, defined by

$$Q_k = \{x \in E : (q_k, x) \in \mathbb{Z}\}.$$

Lemma 4.2. The displacement field is Q_k-invariant: $w_k(x + t) = w_k(x)$ for any t belonging to Q_k. This means that the map φ_k is 'covariant': $\varphi_k(x + t) = \varphi_k(x) + t$ for all t in Q_k.

Proof. If $\boldsymbol{t}$ belongs to Q_k, $(\boldsymbol{q}_k, \boldsymbol{x}+\boldsymbol{t}) = (\boldsymbol{q}_k, \boldsymbol{x}) + z$, where z is an integer; therefore, the fractional parts $r((\boldsymbol{q}_k, \boldsymbol{x}+\boldsymbol{t}))$ and $r((\boldsymbol{q}_k, \boldsymbol{x}))$ are equal and $\boldsymbol{w}_k(\boldsymbol{x}+\boldsymbol{t}) = r((\boldsymbol{q}_k, \boldsymbol{x}+\boldsymbol{t}))\boldsymbol{a}_k^{(k)} = \boldsymbol{w}_k(\boldsymbol{x})$. Finally $\varphi_k(\boldsymbol{x}+\boldsymbol{t}) = \boldsymbol{x}+\boldsymbol{t} - \boldsymbol{w}_k(\boldsymbol{x}+\boldsymbol{t}) = \boldsymbol{x}+\boldsymbol{t} - \boldsymbol{w}_k(\boldsymbol{x}) = \varphi_k(\boldsymbol{x}) + \boldsymbol{t}$. □

Remarks.

(i) In a number of cases the field $\boldsymbol{w}_k$ minimize the amplitude of the displacements necessary to move from $L^{(k-1)}$ to $L^{(k)}$. More precisely the mapping φ_k corresponds to hopping from $L^{(k-1)}$ to the 'nearest' point of $L^{(k)}$ provided the Euclidean metric is replaced (if necessary) by a metric such that $\|\boldsymbol{a}_k^{(k)}/2\| < \|\boldsymbol{a}_i^{(k)} - \boldsymbol{a}_k^{(k)}/2\|$ for $i \neq k$.

(ii) The mapping φ_k is piecewise linear. Indeed, in the strip $\{\boldsymbol{x} \in E : |(\boldsymbol{q}_k, \boldsymbol{x})| < \frac{1}{2}\}$ bounded by two planes parallel to those of Q_k, φ_k coincides with the shear transformation performing the change of basis $(\boldsymbol{a}^{(k-1)}) \to (\boldsymbol{a}^{(k)})$.

(iii) As noticed above, the intersection of $L^{(k-1)}$ and $L^{(k)}$ contains at least the one-dimensional lattice $\mathbb{Z}\boldsymbol{a}_k^{(k)}$. Consequently, the projection of these two lattices onto a hyper-plane perpendicular to $\boldsymbol{a}_k^{(k)}$ give the same $(n-1)$-dimensional lattice. This means that, through each node $\boldsymbol{x}$ of $L^{(k-1)}$, there is a line parallel to $\boldsymbol{a}_k^{(k-1)} = \boldsymbol{a}_k^{(k)}$ which contains the sublattice $\boldsymbol{x} + \mathbb{Z}\boldsymbol{a}_k^{(k-1)}$ of $L^{(k-1)}$ and a similar sublattice $\boldsymbol{y} + \mathbb{Z}\boldsymbol{a}_k^{(k)}$ of $L^{(k)}$. The image $\varphi_k(\boldsymbol{x})$ is simply the point of this second lattice which is the closest to $\boldsymbol{x}$ on the line (see figure 2).

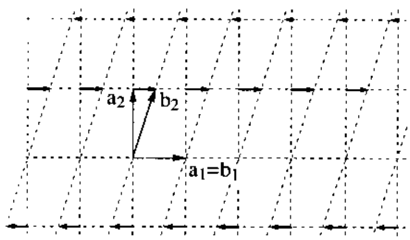

Figure 2. Example of a basic modulation step.

4.2. *The first n stages*

Consider now the mapping of E onto E defined by the composition $\varphi = \varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1$ of the mappings φ_k defined above. Let $Q = \cap Q_k$ denote the intersection of the first n grids Q_k:

$$Q = \{\boldsymbol{x} \in E : (\boldsymbol{q}_k, \boldsymbol{x}) \in \mathbb{Z} \text{ for } k = 1, \ldots, n\}.$$

Generically, i.e. when the vectors $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$ are independent, this intersection is an n-dimensional lattice whose reciprocal basis is $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$. Otherwise, if the rank of $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$ is $r < n$, Q is an r-dimensional periodic array of $(n-r)$-dimensional linear subspaces.

The main properties of the mapping φ are given in the following lemma.

Lemma 4.3. The mapping $\varphi = \varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1$ is one-to-one, close to the identity and the modulation field $\boldsymbol{w} = I - \varphi$ is periodic with respect to Q.

Proof. Since each φ_k is one-to-one and at bounded distance from I, the composition φ is obviously one-to-one and at bounded distance from I.

Assume that t belongs to Q. The covariance of φ follows from the covariance of each of the φ_k (lemma 4.2):

$$\begin{aligned}\varphi_n\circ\ldots\circ\varphi_2\circ\varphi_1(x+t)&=\varphi_n\circ\ldots\circ\varphi_2(\varphi_1(x)+t)=\ldots\\&=\varphi_n(\varphi_{n-1}\circ\ldots\circ\varphi_1(x)+t)\\&=\varphi_n\circ\ldots\circ\varphi_1(x)+t.\end{aligned}$$ □

Since all φ_k have been defined as locally affine maps, so is the global mapping φ. The next two lemmas characterize the domains where φ, and w, are affine.

Lemma 4.4. Let C denote the open domain of E defined by the following conditions

$$C=\{x\in E: |\langle h^{(k)}, x^{(0)}\rangle|<\tfrac{1}{2} \text{ for } k=1,\ldots,n\}$$

where $x^{(0)}$ denotes the coordinates of x in the basis $(a^{(0)})$. Then the restriction of φ to C is linear. (If we set $h^{(k)}=\Sigma_j h_j^{(k)}a_j^{(0)*}$ then C may be defined as $C=\{x\in E: |(h^{(k)}, x)|<\frac{1}{2}$ for $k=1,\ldots,n\}$).

Proof. The first transformation reads $\varphi_1(x)=y$ with

$$y^{(1)}=x^{(0)}+\mathrm{Rnd}(\langle h^{(1)}, x^{(0)}\rangle)e_1$$

where $y^{(1)}$ denotes the coordinates of y in the basis $(a^{(1)})$. If xx belongs to C then we simply have $y^{(1)}=x^{(0)}$, which shows that φ_1 is linear on this domain. Assume that $\varphi_{k-1}\circ\ldots\circ\varphi_2\circ\varphi_1$ is linear on C and is given by $\varphi_{k-1}\circ\ldots\circ\varphi_2\circ\varphi_1(x)=y$ with $y^{(k-1)}=x^{(0)}$, where $y^{(k-1)}$ denotes the coordinates of y in the basis $(a^{(k-1)})$. Then the coordinate set $z^{(k)}$ of $z=\varphi_k(y)$ in the basis $(a^{(k)})$ is given by

$$z^{(k)}=y^{(k-1)}+\mathrm{Rnd}(\langle h^{(k)}, y^{(k-1)}\rangle)e_k.$$

By hypothesis, $\langle h^{(k)}, y^{(k-1)}\rangle=\langle h^{(k)}, x^{(0)}\rangle$ is in the interval $]-\frac{1}{2},\frac{1}{2}[$; consequently, $z^{(k)}=y^{(k-1)}=x^{(0)}$ and $\varphi_k\circ\ldots\circ\varphi_2\circ\varphi_1$ is also linear on C. Finally, the global mapping $\varphi=\varphi_n\circ\ldots\circ\varphi_2\circ\varphi_1$ is linear on C and reads $\varphi(x)=y$ with $y^{(n)}=x^{(0)}$. □

Remark. It is clear, from this proof, that $\varphi|_C$ is the same as the restriction to C of the linear map transforming $(a^{(0)})$ to $(a^{(n)})$.

Lemma 4.5. Assume the set of vectors $\{q_1,\ldots,q_n\}$ is of rank n and let Q and C be as defined above. Then C is a fundamental domain of Q.

Proof. The coordinates q_k of q_k with respect to $a^{(0)*}$ (the dual basis of $a^{(0)}$) are related to $h^{(k)}$ by

$$q_k={}^tM^{(1)}\ldots{}^tM^{(k-1)}h^{(k)}.$$

First we show that $q_k-h^{(k)}$ is a linear combination of $h^{(1)},\ldots,h^{(k-1)}$. Indeed, ${}^tM^{(k-1)}h^{(k)}=h^{(k)}+h_{k-1}^{(k)}h^{(k-1)}$. Next, let j be an integer between 2 and $k-1$ and suppose that ${}^tM^{(j)}\ldots{}^tM^{(k-1)}h^{(k)}-h^{(k)}$ is a linear combination of $h^{(1)},\ldots,h^{(k-1)}$; then the vector

$$\begin{aligned}{}^tM^{(j-1)}\ldots{}^tM^{(k-1)}h^{(k)}-h^{(k)}={}&{}^tM^{(j)}\ldots{}^tM^{(k-1)}h^{(k)}-h^{(k)}\\&+({}^tM^{(j)}\ldots{}^tM^{(k-1)}h^{(k)})_{j-1}h^{(j-1)}\end{aligned}$$

is also a linear combination of $h^{(1)},\ldots,h^{(k-1)}$. By recurrence, this proves that the transition matrix U defined by $q_k=\Sigma_j U_{jk}h^{(j)}$ is in $\mathrm{UT}(n,\mathbb{R})$.

On one hand, this implies that $\{h^{(1)}, \ldots, h^{(n)}\}$ has the same rank as $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$, which is maximal. Hence the set vectors which spans the polytope C—this set is dual to $(\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(n)})$—is a basis of E.

On the other hand, the lattice Q is generated by a basis dual to $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$. As shown in section 5, C is a fundamental domain of Q if the transition matrix U^* relating the basis spanning C to the basis of Q is triangular with ones on the diagonal. But it is the case, here, because $U^* = {}^tU^{-1}$ and the matrix $U: (h^{(1)}, \ldots, h^{(n)}) \to (q_1, \ldots, q_n)$ is in $\mathrm{UT}(n, \mathbb{R})$. □

4.3. *The overall mapping f*

Generically, $Q = \bigcap_{1 \leq k \leq n} Q_k$ is a lattice and C (lemma 4.4) is a non-primitive cell for Q (lemma 4.5). By lemma 4.3, Q is the invariance lattice of $\boldsymbol{w} = I - \varphi$ and C is the linearity domain of $\boldsymbol{w}$ (lemma 4.4).

We now perform the last step φ_{n+1} and prove statements (ii) and (iii) of theorem 1. Under consideration are the overall mapping f defined by

$$f(\boldsymbol{x}) = \varphi_{n+1} \circ \varphi(\boldsymbol{x}) = \varphi_{n+1} \circ \varphi_n \circ \ldots \circ \varphi_1(\boldsymbol{x})$$

and the related displacement field

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{x} - f(\boldsymbol{x}).$$

By proposition 3.4 the last mapping φ_{n+1} is associated to a shear matrix in $\mathcal{M}^{(1)}$; therefore, in $\varphi_{n+1}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{w}_{n+1}(\boldsymbol{x})$, the modulation field is given by $\boldsymbol{w}_{n+1}(\boldsymbol{x}) = r[(\boldsymbol{q}_{n+1}, \boldsymbol{x})]\boldsymbol{a}_1^{(n+1)}$. With $f(\boldsymbol{x}) = \varphi_{n+1}(\boldsymbol{x} - \boldsymbol{w}(\boldsymbol{x})) = \boldsymbol{x} - \boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{w}_{n+1}(\boldsymbol{x} - \boldsymbol{w}(\boldsymbol{x}))$, this implies that the overall modulation is $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x}) + \boldsymbol{w}_{n+1}(\boldsymbol{x} - \boldsymbol{w}(\boldsymbol{x}))$.

Example. For the two square lattices at 45°, f maps the point $\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ of L_a onto the point $\boldsymbol{y} = y_1\boldsymbol{b}_1 + y_2\boldsymbol{b}_2$ of L_b as follows:

$$y_1 = x_1 + \mathrm{Rnd}[(\sqrt{2}-1)x_2)] + \mathrm{Rnd}[(\sqrt{2}-1)(x_2 + \mathrm{Rnd}(-\sqrt{2}(x_1 + \mathrm{Rnd}((\sqrt{2}-1)x_2))/2))]$$

$$y_2 = x_2 + \mathrm{Rnd}[-\sqrt{2}(x_1 + \mathrm{Rnd}((\sqrt{2}-1)x_2))/2].$$

In general, $Q_1 \cap \ldots \cap Q_n \cap Q_{n+1} = Q \cap Q_{n+1}$ reduces to $\{0\}$ so that no translation leaves the vector field $\boldsymbol{v}$ invariant. Nevertheless, notice that $\boldsymbol{w}$ is Q-periodic with reciprocal lattice Q^*, and $\boldsymbol{w}_{n+1}$ is periodic with Fourier spectrum in $Q' = \mathbb{Z}\boldsymbol{q}_{n+1}$; therefore the field $\boldsymbol{v}$, equal to $\boldsymbol{w} + \boldsymbol{w}_{n+1} \circ (I - \boldsymbol{w})$, is quasiperiodic with Fourier spectrum in $Q^* \oplus_Z Q'$.

There is a simple and geometrical way to characterize quasiperiodicity which refers to higher dimension (Besicovich 1932, Oguey *et al* 1988). We apply it here.

Let our Euclidean space E (the 'physical space') be trivially imbedded in the Cartesian product $E \times \mathbb{R} \approx \mathbb{R}^{n+1}$. We shall identify points in E with their imbedding $i(\boldsymbol{x}) = (\boldsymbol{x}, 0) = \boldsymbol{x}$. Set $\boldsymbol{e}_{n+1} = (\boldsymbol{0}, 1)$ and let E' denote the one-dimensional subspace generated by $\boldsymbol{e}_{n+1}$; then $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \boldsymbol{e}_{n+1})$ is a basis of the higher dimensional space $\mathbb{R}^{n+1} \approx E \oplus E'$ and the scalar product in E is the restriction of the scalar product defined by $(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$ for $1 \leq i, j \leq n+1$.

All the maps φ_k, $k = 1, \ldots, n$, as well as the map φ of the previous section extend to 1-1 maps $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting

$$\Phi_k(\xi) = \varphi_k(\boldsymbol{x}) + \boldsymbol{x}'$$

$$\Phi(\xi) = \varphi(\boldsymbol{x}) + \boldsymbol{x}'$$

for all $\xi = x + x' \equiv x + x_{n+1}e_{n+1}$ in $\mathbb{R}^{n+1}$. Correspondingly, the displacement fields w and w_k, $k = 1, \ldots, n$, are extended to E-valued vector fields over $\mathbb{R}^{n+1}$ as

$$W_k(\xi) = W_k(x + x') = w_k(x)$$

$$W(\xi) = W(x + x') = w(x)$$

In this manner, the field W is left invariant by any translation belonging to the bundle of lines $Q \oplus E'$. Similarly, in $(n+1)$ dimensions, the linearity domain for Φ is now the cylinder $C \oplus E'$.

For the last step $k = n+1$ the 'lift' to $\mathbb{R}^{n+1}$ is different: set

$$\theta_{n+1} = q_{n+1} + e_{n+1} (= (q_{n+1}, 1) \text{ in the product notation})$$

and

$$W_{n+1}(\xi) = r((\theta_{n+1}, \xi))a_1^{(n+1)}$$

$$\Phi_{n+1}(\xi) = \xi - W_{n+1}(\xi).$$

Of course, when evaluated on $E = \{\xi \colon \xi = x\}$, that is $\{\xi \colon (\xi, e_{n+1}) = 0\}$, the scalar product reduces to $(\theta_{n+1}, \xi) = (q_{n+1}, x)$ so that Φ_{n+1} and W_{n+1} (restricted to E) coincide with the mappings φ_{n+1} and w_{n+1} defined previously.

The grid

$$\Theta = \{\xi \in \mathbb{R}^{n+1} \colon (\theta_{n+1}, \xi) \in \mathbb{Z}\}.$$

intersects the physical space E on the grid Q_{n+1}. Defined in this way, Φ_{n+1} and W_{n+1} are θ-invariant and affine in the slice

$$\Sigma = \{\xi \in \mathbb{R}^{n+1} \colon -\tfrac{1}{2} < (\eta_{n+1}, \xi) < \tfrac{1}{2}\}$$

where

$$\eta_{n+1} = h_{n+1} + e_{n+1}.$$

So extended to $\mathbb{R}^{n+1}$ the overall displacement map and field

$$F(\xi) = \Phi_{n+1} \circ \Phi(\xi) = \Phi_{n+1}(\Phi(x) + x')$$

$$V(\xi) = \xi - F(\xi)$$

are now covariant (resp. invariant) under translations in the set

$$\Lambda = (Q \oplus E') \cap \Theta$$

which is a regular $(n+1)$-dimensional lattice. If we have at our disposal a basis of Q (we can take the basis (q^*) dual to (q) in E), a basis (ε) for Λ is provided by

$$\varepsilon_k = q_k^* - (q_k^*, q_{n+1})e_{n+1} \qquad k = 1, \ldots, n$$

$$\varepsilon_{n+1} = e_{n+1}.$$

In general, the 'slopes' $-(q_k^*, q_{n+1})$ of E in the basis (ε) are irrational numbers; so E is not a lattice subspace for Λ.

To conclude, as 'cut' by E, by which we mean a restriction to E, of the field V which is periodic in higher dimension (here, $n+1$), the vector field v is quasiperiodic with Fourier components in a $(n+1)$-D module (the projection of Λ^* into E^*).

Moreover, the linearity domain of F and V is

$$\begin{aligned}\Gamma &= (C \oplus E') \cap \Sigma \\ &= \{\xi \in \mathbb{R}^{n+1} \colon |(\eta_{n+1}, \xi)| < \tfrac{1}{2} \text{ and } |(h_k, \xi)| < \tfrac{1}{2} \text{ for } k = 1, \ldots, n\}\end{aligned}$$

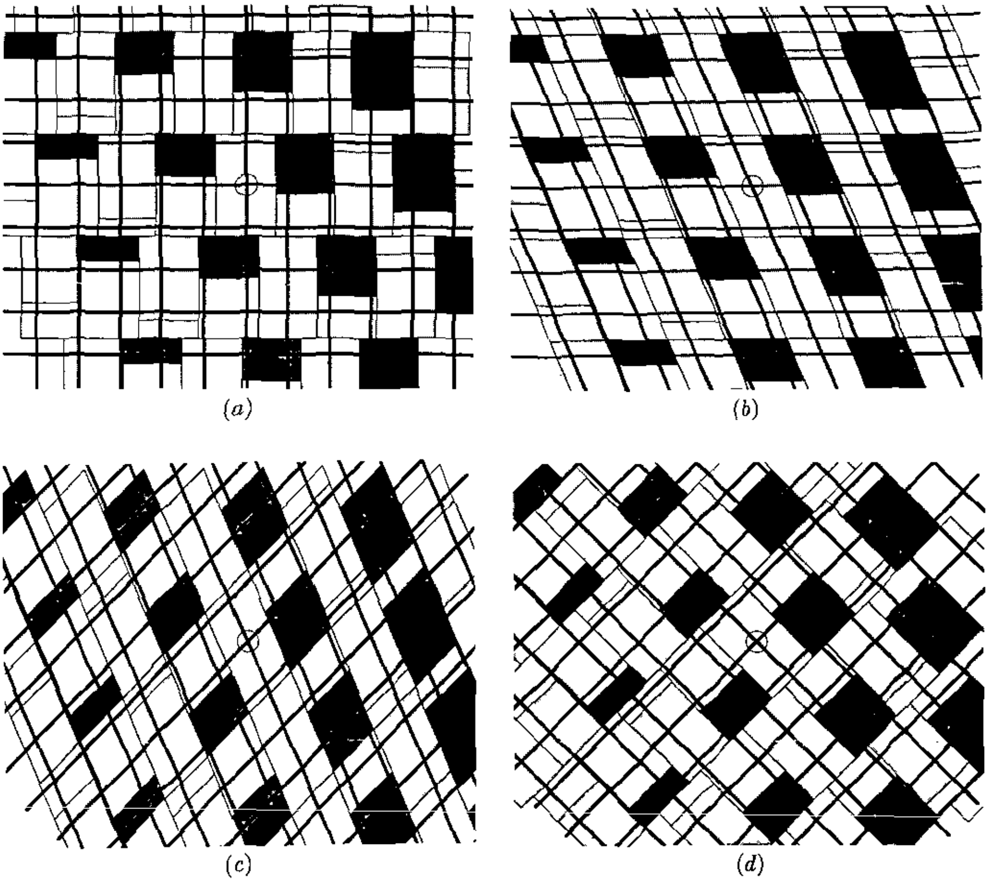

Figure 3. The three basic steps for the square lattices: (a) representation of the initial lattice (bold) and the partition of $\mathbb{R}^2$ into domains where f is affine; (b) the image of (a) by φ_1; (c) is the image of (b) by φ_2; (d) is the image of (c) by φ_3, that is the image of (a) by the modulation f. Each tile stays in the vicinity if its starting position.

which is a non-primitive cell for Λ (proof analogous to lemma 4.4). Cutting this periodic tiling provides a tiling of E which is quasiperiodic, and non-periodic in general (see figure 3). The tiles in the cut are the domains where f and v are affine.

5. Periodic tesselations

In this section, a geometrical interpretation, in terms of lattices, is exhibited of the property, for a matrix, of having its first principal minors equal to one.

Consider the cubic lattice $L_0 = \mathbb{Z}^n$ in $\mathbb{R}^n$ generated by $\{e_1, e_2, \ldots, e_n\}$. The primitive cell is the unit cube $c(n)$ and the space $\mathbb{R}^n$ can be viewed as a packing of non-overlapping cubes:

$$L_0 + c(n) = \bigcup_{x \in L_0} (c(n) + x).$$

This packing is not unique. Indeed, if we let $\boldsymbol{a}_1 = \boldsymbol{e}_1 + \Sigma_{i=2}^n A_{i1}\boldsymbol{e}_i$ where $A_{21}, \ldots, A_{n1}$ are real coefficients, the orbit of $c(n)$ under the lattice generated by $\{\boldsymbol{a}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ provides another packing of $\mathbb{R}^n$ by the same cubes.

Similarly, within each slice of fixed integer coordinate along $\boldsymbol{a}_1$ we can slip co-dimension two slices of cubes with respect to each other; this amounts to letting $\boldsymbol{a}_2 = \boldsymbol{e}_2 + \Sigma_{i=3}^n A_{i2}\boldsymbol{e}_i$, $A_{32}, \ldots, A_{n2}$ real, and disposing the cubes along the sheared lattice generated by $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_n\}$. In the $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ directions, the cubes do not match face to face as they did in the original L_0 packing. This 'locks' those directions for further slipping of the cubes.

Proceeding this way up to n provides a new periodic tesselation of $\mathbb{R}^n$ by cubes set along a lattice L_{a} generated by $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ where

$$\boldsymbol{a}_j = \sum_{i=1}^{n} A_{ij}\boldsymbol{e}_i$$

and the transition matrix A satisfies

$$A_{ii} = 1 \qquad \text{for all } i = 1, \ldots, n$$

$$A_{ij} = 0 \qquad \text{if } 1 \leqslant i < j \leqslant n$$

the second property being due to the 'locking' mechanism. In other words A belongs to the group $\mathrm{LT}(n, \mathbb{R})$.

The order in which the slip directions have been chosen—first perpendicular to $\boldsymbol{e}_1$, next perpendicular to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, then to $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, etc—is of course arbitrary. If P is a permutation matrix and B is any lower triangular matrix then the orbit of $c(n)$ by the lattice L_{b} generated by $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ with $\boldsymbol{b}_i = \Sigma_{j=1}^n ({}^tPBP)_{ji}\boldsymbol{e}_j$ is an equally valid packing of $\mathbb{R}^n$.

What we have just shown is the following: if two lattices L_{a} and L_{b} are related by a transition matrix T of the form

$$T = A^{-1}\,{}^tPBP$$

where P is a permutation and A, B are lower triangular, then they share a common fundamental cell which is a parallelotope (the cube $c(n)$ in the above settings).

Actually, those two conditions—the factorization of the transition matrix and the common cell property—are equivalent.

To show the converse of the above assertion, suppose that a lattice L is given such that $L + c(n)$ is a packing. Then we claim that there is a basis $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ of L such that the transition matrix A from $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ to $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ is triangular, up to permutations (analogous results are in (Hajós 1941). The proof proceeds by recurrence on n. The claim is true for $n = 1$ since, in this case, the lattice is unique and the matrix $A = 1$. Next suppose it is true up to dimension $n - 1$. By the packing hypothesis, the origin 0 lies in the boundary of at least one other cube $c(n) + l$ for some $l \in L$ different from 0. We treat the generic case where 0 belongs to the interior of a facet of $c(n) + l$ (the other cases may be handled by limits of generic cases). By permuting the basis vectors we may assume that this facet is normal to e_1. The non-overlap of $c(n) + l$ and $c(n)$ implies $(\boldsymbol{e}_1, l) = -1$ so that we may define $\boldsymbol{a}_1 = -l = \boldsymbol{e}_1 + \Sigma_{i=2}^n A_{i1}\boldsymbol{e}_i$ where A_{i1} are real numbers, $2 \leqslant i \leqslant n$. In the hyperplane $H = \{x | (\boldsymbol{e}_1, x) = 0\}$ the facet contains a neighbourhood of the origin so that one is left with the problem of covering this neighbourhood with mutually disjoint facets of co-dimension one, that is by cubes $c(n-1)$. But it is not hard to see that the packing in a neighbourhood of a vertex determines the whole periodic packing. Therefore, the plane H is a lattice plane of L. By the recurrence hypothesis, there is a basis $(\boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ of $L \cap H$ connected to $(\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$ by a matrix of $\mathrm{LT}(n-1, \mathbb{R})$ and such that H is tiled by the set of facets $c(n-1)+t$ with t in the lattice generated by $\{\boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$. So if we add $\boldsymbol{a}_1$ to that basis, we get a free set of vectors of L. By construction, the transition matrix $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) \rightarrow (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ is lower triangular; moreover the determinant is one (=volume of $c(n)$), thus $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ is a basis of L.

To conclude, we relate the factorization condition to a simple criterion on the principal minors of the matrix.

Proposition 5.1. Let T be an $n \times n$ matrix. The following are equivalent:

(i) $T = A^{-1}\,{}^t PBP$ where P is a permutation matrix and $A, B \in \mathrm{LT}(n, \mathbb{R})$.

(ii) $\det\{T[(1, \ldots, k)|(1, \ldots, k)]\} = 1$ for all $k = 1, \ldots, n$.

Proof. Let us first show (i)$\Rightarrow$(ii). Note that the diagonal minors of a triangular matrix B (in $\mathrm{LT}(n, \mathbb{R})$) are all equal to one. This is also true for $V = {}^t PBP$ since $\det\{V[p(\alpha)|p(\alpha)]\} = \det\{B[\alpha|\alpha]\}$ for any multi-index α in $\{1, \ldots, n\}$. Now let k be an integer in $\{1, \ldots, n\}$ and set $\alpha = (1, \ldots, k)$. Write $U = A^{-1}$ and V as block matrices

$$U = \begin{pmatrix} U_1 & 0 \\ U_3 & U_4 \end{pmatrix} \qquad V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$$

with $U_1 = U[\alpha|\alpha]$, $V_1 = V[\alpha|\alpha]$ and suitable U_3, U_4, V_2, V_3, V_4. Notice that $U_1 \in \mathrm{LT}(k, \mathbb{R})$. Thus $\det\{UV[\alpha|\alpha]\} = \det\{U_1 V_1\} = \det\{U_1\}\det\{V_1\} = 1$ which proves that $\det\{T[\alpha|\alpha]\} = 1$.

The converse proposition (ii)$\Rightarrow$(i) is a straightforward consequence of the Gauss triangulation procedure (Gantmacher 1977, ch II, section 4). Any square matrix can be represented as a product $T = UDV$ of a lower triangular matrix $U \in \mathrm{LT}(n, \mathbb{R})$, a *diagonal matrix D, and an upper triangular matrix V. The condition (ii) on the principal* minors implies that D is the identity matrix. Moreover, if J is the orthogonal matrix representing the permutation $(1, \ldots, n) \rightarrow (n, n-1, \ldots, 1)$—$J$ provides an isomorphism between $\mathrm{LT}(n, \mathbb{R})$ and $\mathrm{UT}(n, \mathbb{R})$ through the conjugacy $U \rightarrow J^{-1}UJ$—then the matrix $B = {}^t JVJ$ is lower triangular and $T = A^{-1}\,{}^t JBJ$ with $A = U^{-1}$. □

References

Besicovich A S 1932 *Almost Periodic Functions* (Cambridge: Cambridge University Press)

Duneau M and Oguey C 1990 *Euro Phys. Lett.* **13** 67

Gantmacher F R 1977 *The Theory of Matrices* (New York: Chelsea)

Hajós G 1941 *Math. Z.* **47** 427-67

Oguey C, Duneau M and Katz A 1988 *Commun. Math. Phys.* **118** 99

Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951